THE ASYMPTOTIC DISTRIBUTION OF FROBENIUS NUMBERS

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Abstract. The Frobenius number $F(\boldsymbol{a})$ of an integer vector $\boldsymbol{a}$ with positive coprime coefficients is defined as the largest number that does not have a representation as a positive integer linear combination of the coefficients of $\boldsymbol{a}$. We show that if $\boldsymbol{a}$ is taken to be random in an expanding d-dimensional domain, then $F(\boldsymbol{a})$ has a limit distribution, which is given by the probability distribution for the covering radius of a certain simplex with respect to a $(d-1)$-dimensional random lattice. This result extends recent studies for $d = 3$ by Arnold, Bourgain-Sinai and Shur-Sinai-Ustinov. The key features of our approach are (a) a novel interpretation of the Frobenius number in terms of the dynamics of a certain group action on the space of d-dimensional lattices, and (b) an equidistribution theorem for a multidimensional Farey sequence on closed horospheres.

1. Introduction

Let us denote by $\widehat{\mathbb{Z}}^d = \{\boldsymbol{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d : \gcd(a_1, \dots, a_d) = 1\}$ the set of primitive lattice points, and by $\widehat{\mathbb{Z}}^d_{\geq 2}$ the subset with coefficients $a_j \geq 2$. Given $\boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2}$, it is well known that any sufficiently large integer $N > 0$ can be represented in the form

$$N = \boldsymbol{m} \cdot \boldsymbol{a} \tag{1.1}$$

with $\boldsymbol{m} \in \mathbb{Z}^d_{\geq 0}$. Frobenius was interested in the largest integer $F(\boldsymbol{a})$ that fails to have a representation of this type. That is,

$$F(\boldsymbol{a}) = \max \mathbb{Z} \setminus \{\boldsymbol{m} \cdot \boldsymbol{a} > 0 : \boldsymbol{m} \in \mathbb{Z}^d_{\geq 0}\}. \tag{1.2}$$

We will refer to $F(\boldsymbol{a})$ as the *Frobenius number* of $\boldsymbol{a}$. In the case of two variables ($d = 2$) Sylvester showed that

$$F(\boldsymbol{a}) = a_1 a_2 - a_1 - a_2. \tag{1.3}$$

No such explicit formulas are known in higher dimensions, cf. [13], [14], [19]. The present paper will discuss a new interpretation of the Frobenius number in terms of the dynamics of a certain flow Φ^t on the space of lattices $\Gamma\backslash G$, with $G := \mathrm{SL}(d, \mathbb{R})$, $\Gamma := \mathrm{SL}(d, \mathbb{Z})$. This dynamical interpretation is a key step in the proof of the following limit theorem on the asymptotic distribution of the Frobenius number $F(\boldsymbol{a})$, where $\boldsymbol{a}$ is randomly selected from the set $T\mathcal{D} = \{T\boldsymbol{x} : \boldsymbol{x} \in \mathcal{D}\}$, with T large and $\mathcal{D}$ a fixed bounded subset of $\mathbb{R}^d_{\geq 0}$.

Theorem 1. *Let $d \geq 3$. There exists a continuous non-increasing function $\Psi_d : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Psi_d(0) = 1$, such that for any bounded set $\mathcal{D} \subset \mathbb{R}^d_{\geq 0}$ with boundary of Lebesgue measure zero, and any $R \geq 0$,*

$$\lim_{T \to \infty} \frac{1}{T^d} \# \left\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \right\} = \frac{\mathrm{vol}(\mathcal{D})}{\zeta(d)} \, \Psi_d(R). \tag{1.4}$$

Variants of Theorem 1 were previously known only in dimension $d = 3$, cf. [7], [21]; see also [3], [4] for related studies and open conjectures, and [2], [7] for results in higher dimensions. The scaling of $F(\boldsymbol{a})$ used in Theorem 1 is consistent with numerical experiments [5, Section 5].

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We will furthermore establish that the limit distribution $\Psi_d(R)$ is given by the distribution of the covering radius of the simplex

$$\Delta = \{\boldsymbol{x} \in \mathbb{R}_{\geq 0}^{d-1} : \boldsymbol{x} \cdot \boldsymbol{e} \leq 1\}, \qquad \boldsymbol{e} := (1,1,\ldots,1), \tag{1.5}$$

with respect to a random lattice in $\mathbb{R}^{d-1}$. Here, the *covering radius* (sometimes also called *inhomogeneous minimum*) of a set $K \subset \mathbb{R}^{d-1}$ with respect to a lattice $\mathcal{L} \subset \mathbb{R}^{d-1}$ is defined as the infimum of all $\rho > 0$ with the property that $\mathcal{L} + \rho K = \mathbb{R}^{d-1}$.

To state this result precisely, let $\mathbb{Z}^{d-1}A$ be a lattice in $\mathbb{R}^{d-1}$ with $A \in G_0 := \mathrm{SL}(d-1,\mathbb{R})$. The *space of lattices* (of unit covolume) is $\Gamma_0\backslash G_0$ with $\Gamma_0 := \mathrm{SL}(d-1,\mathbb{Z})$. We denote by μ_0 the unique G_0-right invariant probability measure on $\Gamma_0\backslash G_0$; an explicit formula for μ_0 is given in Section 3.

Theorem 2. *Let $\rho(A)$ be the covering radius of the simplex Δ with respect to the lattice $\mathbb{Z}^{d-1}A$. Then*

$$\Psi_d(R) = \mu_0\big(\{A \in \Gamma_0\backslash G_0 : \rho(A) > R\}\big). \tag{1.6}$$

The connection between Frobenius numbers and lattice free simplices is well understood [9], [16]. In particular, Theorem 2 connects nicely to the sharp lower bound of [1] (see also [15]):

$$\frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} \geq \rho_*, \qquad \text{with } \rho_* := \inf_{A \in \Gamma_0\backslash G_0} \rho(A). \tag{1.7}$$

It is proved in [1] that $\rho_* > ((d-1)!)^{1/(d-1)} > 0$, and so in particular

$$\Psi_d(R) = 1 \qquad \text{for} \quad 0 \leq R < \rho_*. \tag{1.8}$$

An explicit formula for $\Psi_d(R)$ has recently been derived in dimension $d = 3$ by different techniques, cf. [21]. In this case $\rho_* = \sqrt{3}$.

It is amusing to note that all of the above statements also hold in the trivial case $d = 2$, except for the continuity of the limit distribution: By Sylvester's formula (1.3)

$$\Psi_2(R) = \begin{cases} 1 & (R < 1) \\ 0 & (R \geq 1). \end{cases} \tag{1.9}$$

The covering radius of the simplex $\Delta = [0,1]$ with respect to the lattice $\mathbb{Z}$ is $\rho(1) = 1$. $\mathbb{Z}$ is of course the unique element in the space of one-dimensional lattices of unit covolume, and hence (1.9) follows also formally from (1.6).

We now give a brief outline of the paper. Section 2 explains the aforementioned dynamical interpretation of the Frobenius number in terms of the right action of a one-parameter subgroup Φ^t on the space of lattices $\Gamma\backslash G$: We show that there is a function W_δ of $\Gamma\backslash G$ that produces, when evaluated along a certain orbit of Φ^t, the Frobenius number $F(\boldsymbol{a})$. This observation is the crucial step in the application of an equidistribution theorem for multidimensional Farey sequences on closed horospheres in $\Gamma\backslash G$, which is proved in Section 3. A useful variant of this theorem is discussed in Section 4. Section 5 exploits the equidistribution theorem to give upper and lower bounds for the lim sup and lim inf of (1.4), respectively, and the purpose of the remaining Sections 6 and 7 is to show that the lim sup and lim inf coincide. This is achieved by relating the limit distribution $\Psi_d(R)$ to the covering radius of a simplex with respect to a random lattice (Section 6), and proving that $\Psi_d(R)$ is continuous (Section 7).

The results of Sections 3 and 4 provide a new approach to Schmidt's work [17] on the distribution of (primitive) sublattices of $\mathbb{Z}^d$. Appendix A illuminates this connection by deriving a generalization of Schmidt's Theorem 3 in the case of primitive sublattices of rank $d-1$.

2. Dynamical interpretation

Let $G := \mathrm{SL}(d,\mathbb{R})$ and $\Gamma := \mathrm{SL}(d,\mathbb{Z})$, and define

$$n_+(\boldsymbol{x}) = \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{x} & 1 \end{pmatrix}, \qquad n_-(\boldsymbol{x}) = \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{x} \\ \boldsymbol{0} & 1 \end{pmatrix}, \qquad \Phi^t = \begin{pmatrix} \mathrm{e}^{-t}1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & \mathrm{e}^{(d-1)t} \end{pmatrix}. \tag{2.1}$$

The right action

$$\Gamma\backslash G \to \Gamma\backslash G, \qquad \Gamma M \mapsto \Gamma M \Phi^t \tag{2.2}$$

defines a flow on the space of lattices $\Gamma\backslash G$. The horospherical subgroups generated by $n_+(\boldsymbol{x})$ and $n_-(\boldsymbol{x})$ parametrize the stable and unstable directions of the flow Φ^t as $t \to \infty$. This can be seen as follows. Let $d : G \times G \to \mathbb{R}_{\geq 0}$ be a left G-invariant Riemannian metric on G, i.e., $d(hM, hM') = d(M, M')$ for all $h, M, M' \in G$. We may choose d in such a way that

$$d\big(n_\pm(\boldsymbol{x}), n_\pm(\boldsymbol{x}')\big) \leq \|\boldsymbol{x} - \boldsymbol{x}'\|, \tag{2.3}$$

where $\|\cdot\|$ the standard euclidean norm. Note that $n_-(\boldsymbol{x})\Phi^t = \Phi^t n_-(\mathrm{e}^{dt}\boldsymbol{x})$. Hence, for any $M \in G$,

$$d\big(M n_-(\boldsymbol{x})\Phi^t, M\Phi^t\big) = d\big(M\Phi^t n_-(\mathrm{e}^{dt}\boldsymbol{x}), M\Phi^t\big) = d\big(n_-(\mathrm{e}^{dt}\boldsymbol{x}), 1_d\big) \leq \mathrm{e}^{dt}\|\boldsymbol{x}\|, \tag{2.4}$$

which explains the interpretation of $n_-(\boldsymbol{x})$ as an element in the *unstable* horospherical subgroup. The argument for $n_+(\boldsymbol{x})$ as the stable analogue is identical.

In the following we will represent functions on $\Gamma\backslash G$ as left Γ-invariant functions on G, i.e., functions $f : G \to \mathbb{R}$ that satisfy $f(\gamma M) = f(M)$ for all $\gamma \in \Gamma$. The left G-invariant metric $d(\,\cdot\,,\,\cdot\,)$ yields thus a Riemannian metric $d_\Gamma(\,\cdot\,,\,\cdot\,)$ on $\Gamma\backslash G$ by setting

$$d_\Gamma(M, M') := \min_{\gamma\in\Gamma} d(M, \gamma M'). \tag{2.5}$$

Indeed, the left G-invariance of d implies $d_\Gamma(\gamma M, M') = d_\Gamma(M, M') = d_\Gamma(M, \gamma M')$ for any $\gamma \in \Gamma$.

The aim of the present section is to identify a function W_δ on $\Gamma\backslash G$ that, when evaluated along a specific orbit of the flow Φ^t, produces the Frobenius number. (As we shall see below, the situation is slightly more complicated in that W_δ also depends on additional variables in $\mathbb{R}^{d-1}$.)

We will assume throughout that $\boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2}$. Following [8], [18] we reduce the Frobenius problem modulo a_d. For $r \in \mathbb{Z}/a_d\mathbb{Z}$ set

$$F_r(\boldsymbol{a}) = \max(r + a_d\mathbb{Z}) \setminus \{\boldsymbol{m}\cdot\boldsymbol{a} > 0 : \boldsymbol{m} \in \mathbb{Z}^d_{\geq 0},\ \boldsymbol{m}\cdot\boldsymbol{a} \equiv r \bmod a_d\} \tag{2.6}$$

Then

$$F(\boldsymbol{a}) = \max_{r \bmod a_d} F_r(\boldsymbol{a}). \tag{2.7}$$

Consider the smallest positive integer that has a representation in $r \bmod a_d$,

$$N_r(\boldsymbol{a}) = \min\{\boldsymbol{m}\cdot\boldsymbol{a} > 0 : \boldsymbol{m} \in \mathbb{Z}^d_{\geq 0},\ \boldsymbol{m}\cdot\boldsymbol{a} \equiv r \bmod a_d\}. \tag{2.8}$$

Then $F_r(\boldsymbol{a}) = N_r(\boldsymbol{a}) - a_d$. We have in fact

$$N_r(\boldsymbol{a}) = \begin{cases} a_d & (r \equiv 0 \bmod a_d) \\ \min\{\boldsymbol{m}'\cdot\boldsymbol{a}' : \boldsymbol{m}' \in \mathbb{Z}^{d-1}_{\geq 0},\ \boldsymbol{m}'\cdot\boldsymbol{a}' \equiv r \bmod a_d\} & (r \not\equiv 0 \bmod a_d) \end{cases} \tag{2.9}$$

with $\boldsymbol{a}' = (a_1, \ldots, a_{d-1})$. In view of (2.7) we conclude

$$F(\boldsymbol{a}) = \max_{r\not\equiv 0 \bmod a_d} N_r(\boldsymbol{a}) - a_d. \tag{2.10}$$

We assume in the following $a_1, \ldots, a_{d-1} \leq a_d \leq T$, and $0 < \delta \leq \frac{1}{2}$. For $r \not\equiv 0 \bmod a_d$ we then have

$$N_r(\boldsymbol{a}) = \min\left\{\boldsymbol{m}'\cdot\boldsymbol{a}' : \boldsymbol{m} \in \mathbb{Z}^{d-1}_{\geq 0} \times \mathbb{Z},\ \big|\boldsymbol{m}\cdot\boldsymbol{a} - r\big| < \frac{\delta a_d}{T}\right\}. \tag{2.11}$$

For $\boldsymbol{\xi}=(\boldsymbol{\xi}',\xi_d)\in\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d$, set

$$N(\boldsymbol{a},\boldsymbol{\xi},T):=\min\nolimits_+\left\{(\boldsymbol{m}'+\boldsymbol{\xi}')\cdot\boldsymbol{a}' : \boldsymbol{m}+\boldsymbol{\xi}\in(\mathbb{Z}^d+\boldsymbol{\xi})\cap\mathbb{R}_{\geq 0}^{d-1}\times\mathbb{R},\ \left|(\boldsymbol{m}+\boldsymbol{\xi})\cdot\boldsymbol{a}\right|<\frac{\delta a_d}{T}\right\}, \tag{2.12}$$

where $\min_+$ is defined by

$$\min\nolimits_+\mathcal{A}=\begin{cases}\min\mathcal{A}\cap\mathbb{R}_{\geq 0} & (\mathcal{A}\cap\mathbb{R}_{\geq 0}\neq\emptyset)\\ 0 & (\mathcal{A}\cap\mathbb{R}_{\geq 0}=\emptyset).\end{cases} \tag{2.13}$$

It is evident that $N(\boldsymbol{a},\boldsymbol{\xi},T)$ is indeed well defined as a function of $\boldsymbol{\xi}\in\mathbb{T}^d$, and furthermore $N_r(\boldsymbol{a})=N(\boldsymbol{a},(\boldsymbol{0},-\frac{r}{a_d}),T)$.

Lemma 1. *Let $\boldsymbol{a}=(a_1,\ldots,a_d)\in\widehat{\mathbb{Z}}_{\geq 2}^d$ with $a_1,\ldots,a_{d-1}\leq a_d\leq T$, $0<\delta\leq\frac12$. Then*

$$F(\boldsymbol{a})=\sup_{\boldsymbol{\xi}\in\mathbb{R}^d/\mathbb{Z}^d}N(\boldsymbol{a},\boldsymbol{\xi},T)-\boldsymbol{e}\cdot\boldsymbol{a}, \tag{2.14}$$

where $\boldsymbol{e}=(1,1,\ldots,1)$.

Proof. Substituting ξ_d by $\xi_d-\boldsymbol{\xi}'\cdot\frac{\boldsymbol{a}'}{a_d}$, we have

$$\begin{aligned}&\sup_{\boldsymbol{\xi}\in\mathbb{R}^d/\mathbb{Z}^d}N(\boldsymbol{a},\boldsymbol{\xi},T)\\ &=\sup_{\substack{\boldsymbol{\xi}'\in[0,1)^{d-1}\\ \xi_d\in\mathbb{T}^1}}\min\nolimits_+\left\{(\boldsymbol{m}'+\boldsymbol{\xi}')\cdot\boldsymbol{a}' : \boldsymbol{m}+\boldsymbol{\xi}\in(\mathbb{Z}^d+\boldsymbol{\xi})\cap\mathbb{R}_{\geq 0}^{d-1}\times\mathbb{R},\ \left|\boldsymbol{m}\cdot\boldsymbol{a}+\xi_d a_d\right|<\frac{\delta a_d}{T}\right\}\\ &=\sup_{\substack{\boldsymbol{\xi}'\in[0,1)^{d-1}\\ \xi_d\in\mathbb{T}^1}}\min\nolimits_+\left\{(\boldsymbol{m}'+\boldsymbol{\xi}')\cdot\boldsymbol{a}' : \boldsymbol{m}\in\mathbb{Z}_{\geq 0}^{d-1}\times\mathbb{Z},\ \left|\boldsymbol{m}\cdot\boldsymbol{a}+\xi_d a_d\right|<\frac{\delta a_d}{T}\right\}\\ &=\sup_{\xi_d\in\mathbb{T}^1}\min\nolimits_+\left\{\boldsymbol{m}'\cdot\boldsymbol{a}' : \boldsymbol{m}\in\mathbb{Z}_{\geq 0}^{d-1}\times\mathbb{Z},\ \left|\boldsymbol{m}\cdot\boldsymbol{a}+\xi_d a_d\right|<\frac{\delta a_d}{T}\right\}+\boldsymbol{e}\cdot\boldsymbol{a}',\end{aligned} \tag{2.15}$$

where $\boldsymbol{e}=(1,1,\ldots,1)$. The second equality follows from the fact that for $1\leq j<d$, $m_j+\xi_j\geq 0$ implies $m_j\geq 0$ since $m_j\in\mathbb{Z}$ and $\xi_j\in[0,1)$. We observe that, since $\frac{\delta a_d}{T}\leq\frac12$ and $\boldsymbol{m}\cdot\boldsymbol{a}\in\mathbb{Z}$, we can replace in the inequality $|\boldsymbol{m}\cdot\boldsymbol{a}+\xi_d a_d|<\frac{\delta a_d}{T}$ the quantity $\xi_d a_d$ by its nearest integer, say s. That is, (2.15) equals

$$\sup_{s \bmod a_d}\min\nolimits_+\left\{\boldsymbol{m}'\cdot\boldsymbol{a}' : \boldsymbol{m}\in\mathbb{Z}_{\geq 0}^{d-1}\times\mathbb{Z},\ \left|\boldsymbol{m}\cdot\boldsymbol{a}+s\right|<\frac{\delta a_d}{T}\right\}+\boldsymbol{e}\cdot\boldsymbol{a}'. \tag{2.16}$$

The case $s\equiv 0 \bmod a_d$ does not contribute (because then $\boldsymbol{m}=\boldsymbol{0}$ achieves 0 as minimum). Since $0\leq a_j\leq a_d$ we thus obtain

$$\max_{r\not\equiv 0 \bmod a_d}N_r(\boldsymbol{a})=\sup_{\boldsymbol{\xi}\in\mathbb{R}^d/\mathbb{Z}^d}N(\boldsymbol{a},\boldsymbol{\xi},T)-\boldsymbol{e}\cdot\boldsymbol{a}', \tag{2.17}$$

and the lemma follows from (2.10). □

Let W_δ denote the function $\mathbb{R}_{\geq 0}^{d-1}\times G\to\mathbb{R}$, $(\boldsymbol{\alpha},M)\mapsto W_\delta(\boldsymbol{\alpha},M)$, given by

$$W_\delta(\boldsymbol{\alpha},M)=\sup_{\boldsymbol{\xi}\in\mathbb{T}^d}\min\nolimits_+\left\{(\boldsymbol{m}+\boldsymbol{\xi})M\cdot(\boldsymbol{\alpha},0) : \boldsymbol{m}\in\mathbb{Z}^d,\ (\boldsymbol{m}+\boldsymbol{\xi})M\in\mathcal{R}_\delta\right\} \tag{2.18}$$

where $\mathcal{R}_\delta=\mathbb{R}_{\geq 0}^{d-1}\times(-\delta,\delta)$. Note that for every $\gamma\in\Gamma$

$$\begin{aligned}W_\delta(\boldsymbol{\alpha},\gamma M)&=\sup_{\boldsymbol{\xi}\in\mathbb{T}^d}\min\nolimits_+\left\{(\boldsymbol{m}+\boldsymbol{\xi})\gamma M\cdot(\boldsymbol{\alpha},0) : \boldsymbol{m}\in\mathbb{Z}^d,\ (\boldsymbol{m}+\boldsymbol{\xi})\gamma M\in\mathcal{R}_\delta\right\}\\ &=\sup_{\boldsymbol{\xi}\in\mathbb{T}^d\gamma}\min\nolimits_+\left\{(\boldsymbol{m}+\boldsymbol{\xi})M\cdot(\boldsymbol{\alpha},0) : \boldsymbol{m}\in\mathbb{Z}^d\gamma,\ (\boldsymbol{m}+\boldsymbol{\xi})M\in\mathcal{R}_\delta\right\}.\end{aligned} \tag{2.19}$$

Both $\mathbb{Z}^d$ and $\mathbb{T}^d$ are Γ-invariant; thus

$$W_\delta(\boldsymbol{\alpha}, \gamma M) = W_\delta(\boldsymbol{\alpha}, M) \tag{2.20}$$

for all $\boldsymbol{\alpha} \in \mathbb{R}_{\geq 0}^{d-1}$, $M \in G$ and $\gamma \in \Gamma$.

Combining Definition (2.18) with Lemma 1 (set $t = \frac{\log T}{d-1}$) we obtain:

Theorem 3. *Let $\boldsymbol{a} = (a_1, \ldots, a_d) \in \widehat{\mathbb{Z}}_{\geq 2}^d$ with $a_1, \ldots, a_{d-1} \leq a_d \leq \mathrm{e}^{(d-1)t}$, and $0 < \delta \leq \frac{1}{2}$. Then*

$$F(\boldsymbol{a}) = \mathrm{e}^t W_\delta\big(\boldsymbol{a}', n_-(\widehat{\boldsymbol{a}})\Phi^t\big) - \boldsymbol{e} \cdot \boldsymbol{a}, \tag{2.21}$$

where

$$\widehat{\boldsymbol{a}} = \frac{\boldsymbol{a}'}{a_d} = \left(\frac{a_1}{a_d}, \ldots, \frac{a_{d-1}}{a_d}\right). \tag{2.22}$$

3. Farey sequences on horospheres

Denote by $\mu = \mu_G$ the Haar measure on $G = \mathrm{SL}(d, \mathbb{R})$, normalized so that it represents the unique right G-invariant probability measure on the homogeneous space $\Gamma\backslash G$, where $\Gamma = \mathrm{SL}(d, \mathbb{Z})$. By Siegel's volume formula

$$d\mu(M) \frac{dt}{t} = \big(\zeta(2)\zeta(3)\cdots\zeta(d)\big)^{-1} \det(X)^{-d} \prod_{i,j=1}^{d} dX_{ij}, \tag{3.1}$$

where $X = (X_{ij}) = t^{1/d} M \in \mathrm{GL}^+(d, \mathbb{R})$ with $M \in G$, $t > 0$, cf. [10], [22]. We will also use the notation μ_0 for the right G_0-invariant probability measure on $\Gamma_0\backslash G_0$, with $G_0 = \mathrm{SL}(d-1, \mathbb{R})$ and $\Gamma_0 = \mathrm{SL}(d-1, \mathbb{Z})$.

Consider the subgroups

$$H = \Big\{ M \in G : (\boldsymbol{0}, 1) M = (\boldsymbol{0}, 1) \Big\} = \left\{ \begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} : A \in G_0,\ \boldsymbol{b} \in \mathbb{R}^{d-1} \right\} \tag{3.2}$$

and

$$\Gamma_H = \Gamma \cap H = \left\{ \begin{pmatrix} \gamma & {}^{\mathrm{t}}\boldsymbol{m} \\ \boldsymbol{0} & 1 \end{pmatrix} : \gamma \in \Gamma_0,\ \boldsymbol{m} \in \mathbb{Z}^{d-1} \right\}. \tag{3.3}$$

Note that H and Γ_H are isomorphic to $\mathrm{ASL}(d-1, \mathbb{R})$ and $\mathrm{ASL}(d-1, \mathbb{Z})$, respectively. We normalize the Haar measure μ_H of H so that it becomes a probability measure on $\Gamma_H\backslash H$; explicitly:

$$d\mu_H(M) = d\mu_0(A)\, d\boldsymbol{b}, \qquad M = \begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix}. \tag{3.4}$$

The following states the classical equidistribution theorem for Φ^t-translates of the closed horospheres $\Gamma\backslash\Gamma\{n_-(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{T}^{d-1}\}$ on $\Gamma\backslash G$; cf. [11, Section 5].

Theorem 4. *Let λ be a Borel probability measure on $\mathbb{T}^{d-1}$, absolutely continuous with respect to Lebesgue measure, and let $f : \mathbb{T}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ be bounded continuous. Then*

$$\lim_{t\to\infty} \int_{\mathbb{T}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big)\, d\lambda(\boldsymbol{x}) = \int_{\mathbb{T}^{d-1}\times\Gamma\backslash G} f(\boldsymbol{x}, M)\, d\lambda(\boldsymbol{x})\, d\mu(M). \tag{3.5}$$

A standard probabilistic argument [20, Chapter III] allows to reformulate the above statement in terms characteristic functions of subsets of $\mathbb{T}^{d-1} \times \Gamma\backslash G$.

Theorem 5. *Take λ as in Theorem 4, and let $\mathcal{A} \subset \mathbb{T}^{d-1} \times \Gamma\backslash G$. Then*

$$\liminf_{t\to\infty} \lambda\big(\big\{\boldsymbol{x} \in \mathbb{T}^{d-1} : \big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) \in \mathcal{A}\big\}\big) \geq (\lambda \times \mu)(\mathcal{A}^\circ) \tag{3.6}$$

and

$$\limsup_{t\to\infty} \lambda\big(\big\{\boldsymbol{x} \in \mathbb{T}^{d-1} : \big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) \in \mathcal{A}\big\}\big) \leq (\lambda \times \mu)(\overline{\mathcal{A}}). \tag{3.7}$$

Remark 3.1. This shows that Theorem 4 can be extended to test functions f that are characteristic functions of subsets of $\mathbb{T}^{d-1} \times \Gamma\backslash G$ with boundary of $(\lambda \times \mu)$-measure zero [11, Sect. 5.3], and thus also to functions that are the product of such a characteristic function and a bounded continuous function.

We will now replace the absolutely continuous measure λ by equally weighted point masses at the elements of the Farey sequence

$$\mathcal{F}_Q = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^{d-1} : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^d,\ 0 < q \leq Q \right\}, \tag{3.8}$$

for $Q \in \mathbb{N}$. Note that

$$|\mathcal{F}_Q| \sim \frac{Q^d}{d\,\zeta(d)} \qquad (Q \to \infty). \tag{3.9}$$

It will be notationally convenient to also allow general $Q \in \mathbb{R}_{\geq 1}$ in the definition (3.8) of $\mathcal{F}_Q$; note that $\mathcal{F}_Q = \mathcal{F}_{[Q]}$ where $[Q]$ is the integer part of Q.

Theorem 6. *Fix $\sigma \in \mathbb{R}$. Let $f : \mathbb{T}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ be bounded continuous. Then, for $Q = \mathrm{e}^{(d-1)(t-\sigma)}$,*

$$\lim_{t\to\infty} \frac{1}{|\mathcal{F}_Q|} \sum_{\boldsymbol{r} \in \mathcal{F}_Q} f\big(\boldsymbol{r}, n_-(\boldsymbol{r})\Phi^t\big) = d(d-1)\mathrm{e}^{d(d-1)\sigma} \int_\sigma^\infty \int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H} \widetilde{f}(\boldsymbol{x}, M\Phi^{-s})\, d\boldsymbol{x}\, d\mu_H(M)\, \mathrm{e}^{-d(d-1)s} ds \tag{3.10}$$

with $\widetilde{f}(\boldsymbol{x}, M) := f(\boldsymbol{x}, {}^{\mathrm{t}}M^{-1})$.

Remark 3.2. The identical argument as in Remark 3.1 permits the extension of Theorem 6 to any test function f which is the product of a bounded continuous function and the characteristic function of a subset $\mathcal{A} \subset \mathbb{T}^{d-1} \times \Gamma\backslash G$, where $\widetilde{\mathcal{A}} = \{(\boldsymbol{x}, M) : (\boldsymbol{x}, {}^{\mathrm{t}}M^{-1}) \in \mathcal{A}\}$ has boundary of measure zero with respect to $d\boldsymbol{x}\, d\mu_H\, ds$.

Proof of Theorem 6. **Step 0: Uniform continuity.** By choosing the test function $f(\boldsymbol{x}, M) = f_0(\boldsymbol{x}, M\Phi^{-\sigma})$ with $f_0 : \mathbb{T}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ bounded continuous, it is evident that we only need to consider the case $\sigma = 0$. We may also assume without loss of generality that f, and thus $\widetilde{f}$, have compact support. That is, there is $\mathcal{C} \subset G$ compact such that $\operatorname{supp} f, \operatorname{supp} \widetilde{f} \subset \mathbb{T}^{d-1} \times \Gamma\backslash\Gamma\mathcal{C}$. The generalization to bounded continuous functions follows from a standard approximation argument.

Since f is continuous and has compact support, it is uniformly continuous. That is, given any $\delta > 0$ there exists $\epsilon > 0$ such that for all $(\boldsymbol{x}, M), (\boldsymbol{x}', M') \in \mathbb{R}^{d-1} \times G$,

$$\|\boldsymbol{x} - \boldsymbol{x}'\| < \epsilon, \qquad d(M, M') < \epsilon \tag{3.11}$$

implies

$$\big|f(\boldsymbol{x}, M) - f(\boldsymbol{x}', M')\big| < \delta. \tag{3.12}$$

The plan is now to first establish (3.10) for the set

$$\mathcal{F}_{Q,\theta} = \left\{ \frac{\boldsymbol{p}}{q} \in [0,1)^{d-1} : (\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}^d,\ \theta Q < q \leq Q \right\}, \tag{3.13}$$

for any $\theta \in (0,1)$. The constant θ will remain fixed until the very last step of this proof.

Step 1: Thicken the Farey sequence. The plan is to reduce the statement to Theorem 4. To this end, we thicken the set $\mathcal{F}_{Q,\theta}$ as follows: For $\epsilon > 0$ (we will in fact later use the ϵ from Step 0), let

$$\mathcal{F}_Q^\epsilon = \bigcup_{\boldsymbol{r} \in \mathcal{F}_{Q,\theta} + \mathbb{Z}^{d-1}} \big\{ \boldsymbol{x} \in \mathbb{R}^{d-1} : \|\boldsymbol{x} - \boldsymbol{r}\| < \epsilon \mathrm{e}^{-dt} \big\}. \tag{3.14}$$

Note that $\mathcal{F}_Q^\epsilon$ is symmetric with respect to $\boldsymbol{x} \mapsto -\boldsymbol{x}$. A short calculation yields

$$\mathcal{F}_Q^\epsilon = \bigcup_{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d} \big\{\boldsymbol{x} \in \mathbb{R}^{d-1} : \boldsymbol{a}\, n_+(\boldsymbol{x})\Phi^{-t} \in \mathfrak{C}_\epsilon\big\}, \tag{3.15}$$

where

$$\mathfrak{C}_\epsilon = \big\{(y_1,\ldots,y_d) \in \mathbb{R}^d : \|(y_1,\ldots,y_{d-1})\| < \epsilon y_d,\ \theta < y_d \le 1\big\}. \tag{3.16}$$

Let

$$\mathcal{H}_\epsilon = \bigcup_{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d} \mathcal{H}_\epsilon(\boldsymbol{a}), \qquad \mathcal{H}_\epsilon(\boldsymbol{a}) = \big\{M \in G : \boldsymbol{a}M \in \mathfrak{C}_\epsilon\big\}. \tag{3.17}$$

The bijection (cf. [22])

$$\Gamma_H\backslash\Gamma \to \widehat{\mathbb{Z}}^d, \qquad \Gamma_H\gamma \mapsto (\boldsymbol{0},1)\gamma \tag{3.18}$$

allows us to rewrite

$$\mathcal{H}_\epsilon = \bigcup_{\gamma\in\Gamma_H\backslash\Gamma} \mathcal{H}_\epsilon((\boldsymbol{0},1)\gamma) = \bigcup_{\gamma\in\Gamma/\Gamma_H} \gamma\mathcal{H}_\epsilon^1, \qquad \text{with } \mathcal{H}_\epsilon^1 = \mathcal{H}_\epsilon((\boldsymbol{0},1)). \tag{3.19}$$

Now

$$\begin{aligned}\mathcal{H}_\epsilon^1 =&\big\{M \in G : (\boldsymbol{0},1)M \in \mathfrak{C}_\epsilon\big\}\\ =&H\big\{M_{\boldsymbol{y}} : \boldsymbol{y} \in \mathfrak{C}_\epsilon\big\}\end{aligned} \tag{3.20}$$

with H as in (3.2), and $M_{\boldsymbol{y}} \in G$ such that $(\boldsymbol{0},1)M_{\boldsymbol{y}} = \boldsymbol{y}$. Since $\boldsymbol{y} \in \mathfrak{C}_\epsilon$ implies $y_d > 0$, we may choose

$$M_{\boldsymbol{y}} = \begin{pmatrix} y_d^{-1/(d-1)} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{y}' & y_d \end{pmatrix}, \qquad \boldsymbol{y}' = (y_1,\ldots,y_{d-1}). \tag{3.21}$$

Step 2: Prove disjointness. We will now prove the following claim: *Given a compact subset $\mathcal{C} \subset G$, there exists $\epsilon_0 > 0$ such that*

$$\gamma\mathcal{H}_\epsilon^1 \cap \mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C} = \emptyset \tag{3.22}$$

for every $\epsilon \in (0,\epsilon_0]$, $\gamma \in \Gamma \setminus \Gamma_H$.

To prove this claim, note that (3.22) is equivalent to

$$\mathcal{H}_\epsilon((\boldsymbol{p},q)) \cap \mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C} = \emptyset \tag{3.23}$$

for every $(\boldsymbol{p},q) \in \widehat{\mathbb{Z}}^d$, $(\boldsymbol{p},q) \neq (\boldsymbol{0},1)$. For

$$M = \begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\boldsymbol{y}}, \qquad M_{\boldsymbol{y}} = \begin{pmatrix} y_d^{-1/(d-1)} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{y}' & y_d \end{pmatrix}, \tag{3.24}$$

we have

$$(\boldsymbol{p},q)M = (\boldsymbol{p}Ay_d^{-1/(d-1)} + (\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)\boldsymbol{y}', (\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d), \tag{3.25}$$

and thus $M \in \mathcal{H}_\epsilon((\boldsymbol{p},q)) \cap \mathcal{H}_\epsilon^1$ if and only if

$$\|\boldsymbol{p}Ay_d^{-1/(d-1)} + (\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)\boldsymbol{y}'\| < \epsilon(\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d, \tag{3.26}$$

$$\theta < (\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d \le 1, \tag{3.27}$$

and

$$\|\boldsymbol{y}'\| < \epsilon y_d, \qquad \theta < y_d \le 1. \tag{3.28}$$

Relations (3.27) and (3.28) imply $\|(\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)\boldsymbol{y}'\| < \epsilon(\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d \le \epsilon$ and so, by (3.26), $\|\boldsymbol{p}Ay_d^{-1/(d-1)}\| < 2\epsilon(\boldsymbol{p}\,{}^{\mathrm{t}}\boldsymbol{b} + q)y_d \le 2\epsilon$. That is, $\|\boldsymbol{p}A\| < 2\epsilon y_d^{1/(d-1)}$ and hence

$$\|\boldsymbol{p}A\| < 2\epsilon. \tag{3.29}$$

Let us now suppose $M \in \Gamma\mathcal{C}$ with $\mathcal{C}$ compact. The set

$$\mathcal{C}' = \mathcal{C}\{M_{\boldsymbol{y}}^{-1} : \boldsymbol{y} \in \overline{\mathfrak{C}}_\epsilon\} \tag{3.30}$$

is still compact, by the compactness of $\overline{\mathfrak{C}}_\epsilon$ (the closure of $\mathfrak{C}_\epsilon$) in $\mathbb{R}^d \setminus \{\mathbf{0}\}$. In view of (3.24) we obtain

$$\begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \mathbf{0} & 1 \end{pmatrix} \in \Gamma\mathcal{C}', \tag{3.31}$$

and so $A \in \Gamma_0\mathcal{C}_0$ for some compact $\mathcal{C}_0 \subset G_0$.

Mahler's compactness criterion then shows that

$$I := \inf_{A \in \Gamma_0\mathcal{C}_0} \inf_{\boldsymbol{p} \in \mathbb{Z}^{d-1}\setminus\{\mathbf{0}\}} \|\boldsymbol{p}A\| > 0. \tag{3.32}$$

Now choose ϵ_0 such that $0 < 2\epsilon_0 < I$. Then (3.29) implies $\boldsymbol{p} = \mathbf{0}$ and therefore $q = 1$. The claim is proved.

Step 3: Apply Theorem 4. Step 2 implies that, for $\mathcal{C} \subset G$ compact, there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0]$

$$\mathcal{H}_\epsilon \cap \Gamma\mathcal{C} = \bigcup_{\gamma \in \Gamma/\Gamma_H} \big(\gamma\mathcal{H}_\epsilon^1 \cap \Gamma\mathcal{C}\big) \tag{3.33}$$

is a disjoint union. Hence, if χ_ϵ and χ_ϵ^1 are the characteristic functions of the sets $\mathcal{H}_\epsilon$ and $\mathcal{H}_\epsilon^1$, respectively, we have

$$\chi_\epsilon(M) = \sum_{\gamma \in \Gamma_H\setminus\Gamma} \chi_\epsilon^1(\gamma M), \tag{3.34}$$

for all $M \in \Gamma\mathcal{C}$. Evidently $\mathcal{H}_\epsilon^1$ and thus $\mathcal{H}_\epsilon$ have boundary of μ-measure zero. We furthermore set $\widetilde{\chi}_\epsilon(M) := \chi_\epsilon({}^{\mathrm{t}}M^{-1})$, and note that $\chi_\epsilon\big(n_+(\boldsymbol{x})\Phi^{-t}\big) = \chi_\epsilon\big(n_+(-\boldsymbol{x})\Phi^{-t}\big)$ is the characteristic function of the set $\mathcal{F}_Q^\epsilon$; recall (3.15) and the remark after (3.14). Therefore

$$\begin{aligned} \int_{\mathcal{F}_Q^\epsilon/\mathbb{Z}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) d\boldsymbol{x} &= \int_{\mathbb{T}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) \chi_\epsilon\big(n_+(-\boldsymbol{x})\Phi^{-t}\big) d\boldsymbol{x} \\ &= \int_{\mathbb{T}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) \widetilde{\chi}_\epsilon\big(n_-(\boldsymbol{x})\Phi^t\big) d\boldsymbol{x}, \end{aligned} \tag{3.35}$$

and Theorem 4 yields

$$\begin{aligned} \lim_{t\to\infty} \int_{\mathbb{T}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) \widetilde{\chi}_\epsilon\big(n_-(\boldsymbol{x})\Phi^t\big)\, d\boldsymbol{x} &= \int_{\mathbb{T}^{d-1}\times\Gamma\setminus G} f(\boldsymbol{x}, M)\widetilde{\chi}_\epsilon(M)\, d\boldsymbol{x}\, d\mu(M) \\ &= \int_{\mathbb{T}^{d-1}\times\Gamma\setminus G} \widetilde{f}(\boldsymbol{x}, M)\chi_\epsilon(M)\, d\boldsymbol{x}\, d\mu(M). \end{aligned} \tag{3.36}$$

Step 4: A volume computation. To evaluate the right hand side of (3.36), we use (3.34):

$$\begin{aligned} \int_{\mathbb{T}^{d-1}\times\Gamma\setminus G} \widetilde{f}(\boldsymbol{x}, M)\chi_\epsilon(M)\, d\boldsymbol{x}\, d\mu(M) &= \int_{\mathbb{T}^{d-1}\times\Gamma_H\setminus G} \widetilde{f}(\boldsymbol{x}, M)\chi_\epsilon^1(M)\, d\boldsymbol{x}\, d\mu(M) \\ &= \int_{\mathbb{T}^{d-1}\times\Gamma_H\setminus\mathcal{H}_\epsilon^1} \widetilde{f}(\boldsymbol{x}, M)\, d\boldsymbol{x}\, d\mu(M). \end{aligned} \tag{3.37}$$

Given $\boldsymbol{y} \in \mathbb{R}^d$ we pick a matrix $M_{\boldsymbol{y}} \in G$ such that $(\mathbf{0}, 1)M_{\boldsymbol{y}} = \boldsymbol{y}$; recall (3.21) for an explicit choice of $M_{\boldsymbol{y}}$ for $y_d > 0$. The map

$$H \times \mathbb{R}^d \setminus \{\mathbf{0}\} \to G, \qquad (M, \boldsymbol{y}) \mapsto MM_{\boldsymbol{y}}, \tag{3.38}$$

provides a parametrization of G, where in view of (3.1)

$$d\mu = \zeta(d)^{-1} d\mu_H\, d\boldsymbol{y}. \tag{3.39}$$

Hence (3.37) equals

$$\frac{1}{\zeta(d)} \int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H\times\mathfrak{C}_\epsilon} \widetilde{f}\big(\boldsymbol{x}, MM_{\boldsymbol{y}}\big)\, d\boldsymbol{x}\, d\mu_H(M) d\boldsymbol{y}. \tag{3.40}$$

For

$$D(y_d) = \begin{pmatrix} y_d^{-1/(d-1)} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & y_d \end{pmatrix}, \tag{3.41}$$

we have

$$d\big(M_{\boldsymbol{y}}, D(y_d)\big) = d\big(D(y_d) n_+(y_d^{-1}\boldsymbol{y}'), D(y_d)\big) = d\big(n_+(y_d^{-1}\boldsymbol{y}'), 1_d\big) \le y_d^{-1}\|\boldsymbol{y}'\|. \tag{3.42}$$

We recall that $y_d^{-1}\|\boldsymbol{y}'\| < \epsilon$ for $\boldsymbol{y} \in \mathfrak{C}_\epsilon$. Therefore, with the choice of δ, ϵ made in Steps 0 and 2, we have (note that (3.12) applies also to $\widetilde{f}$)

$$\left| (3.40) - \frac{1}{\zeta(d)} \int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H\times\mathfrak{C}_\epsilon} \widetilde{f}\big(\boldsymbol{x}, MD(y_d)\big)\, d\boldsymbol{x}\, d\mu_H(M) d\boldsymbol{y} \right| < \frac{\delta}{\zeta(d)} \int_{\mathfrak{C}_\epsilon} d\boldsymbol{y}. \tag{3.43}$$

We have

$$\begin{aligned} \int_{\mathfrak{C}_\epsilon} \widetilde{f}\big(\boldsymbol{x}, MD(y_d)\big)\, d\boldsymbol{y} &= \operatorname{vol}(\mathcal{B}_1^{d-1})\, \epsilon^{d-1} \int_\theta^1 \widetilde{f}\big(\boldsymbol{x}, MD(y_d)\big)\, y_d^{d-1}\, dy_d \\ &= (d-1) \operatorname{vol}(\mathcal{B}_1^{d-1})\, \epsilon^{d-1} \int_0^{|\log\theta|/(d-1)} \widetilde{f}\big(\boldsymbol{x}, M\Phi^{-s}\big)\, \mathrm{e}^{-d(d-1)s} ds, \end{aligned} \tag{3.44}$$

and

$$\int_{\mathfrak{C}_\epsilon} d\boldsymbol{y} = \frac{1}{d} \operatorname{vol}(\mathcal{B}_1^{d-1})\, \epsilon^{d-1} (1-\theta^d), \tag{3.45}$$

where $\mathcal{B}_1^{d-1}$ denotes the unit ball in $\mathbb{R}^{d-1}$. So (3.43) becomes

$$\begin{aligned} &\left| (3.40) - \frac{(d-1)\operatorname{vol}(\mathcal{B}_1^{d-1})\, \epsilon^{d-1}}{\zeta(d)} \int_0^{|\log\theta|/(d-1)} \int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H} \widetilde{f}\big(\boldsymbol{x}, M\Phi^{-s}\big)\, d\boldsymbol{x}\, d\mu_H(M)\, \mathrm{e}^{-d(d-1)s} ds \right| \\ &\qquad < \frac{\operatorname{vol}(\mathcal{B}_1^{d-1})\, \delta\, \epsilon^{d-1}}{d\,\zeta(d)} (1-\theta^d). \end{aligned} \tag{3.46}$$

Step 5: Distance estimates. Since (3.33) is a disjoint union, we have furthermore (this is in effect another way of writing (3.35) using (3.34))

$$\int_{\mathcal{F}_Q^\epsilon/\mathbb{Z}^{d-1}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) d\boldsymbol{x} = \sum_{\boldsymbol{r}\in\mathcal{F}_{Q,\theta}} \int_{\|\boldsymbol{x}-\boldsymbol{r}\|<\epsilon\mathrm{e}^{-dt}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) d\boldsymbol{x}. \tag{3.47}$$

Eq. (2.4) implies that

$$d\big(n_-(\boldsymbol{x})\Phi^t, n_-(\boldsymbol{r})\Phi^t\big) \le \mathrm{e}^{dt}\|\boldsymbol{x}-\boldsymbol{r}\| < \epsilon. \tag{3.48}$$

Because f is uniformly continuous we therefore have, for the same δ, ϵ as above:

$$\left| \int_{\|\boldsymbol{x}-\boldsymbol{r}\|<\epsilon\mathrm{e}^{-dt}} f\big(\boldsymbol{x}, n_-(\boldsymbol{x})\Phi^t\big) d\boldsymbol{x} - \frac{\operatorname{vol}(\mathcal{B}_1^{d-1})\epsilon^{d-1}}{\mathrm{e}^{d(d-1)t}} f\big(\boldsymbol{r}, n_-(\boldsymbol{r})\Phi^t\big) \right| < \frac{\operatorname{vol}(\mathcal{B}_1^{d-1})\, \delta\, \epsilon^{d-1}}{\mathrm{e}^{d(d-1)t}}, \tag{3.49}$$

uniformly for all $t \ge 0$.

Step 6: Conclusion. The approximations (3.46) and (3.49) hold uniformly for any $\delta > 0$. Passing to the limit $\delta \to 0$, we obtain

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{\mathrm{e}^{d(d-1)t}} \sum_{\boldsymbol{r}\in\mathcal{F}_{Q,\theta}} f\big(\boldsymbol{r}, n_-(\boldsymbol{r})\Phi^t\big) \\ &\qquad = \frac{d-1}{\zeta(d)} \int_0^{|\log\theta|/(d-1)} \int_{\mathbb{T}^{d-1}\times\Gamma_H\backslash H} \widetilde{f}\big(\boldsymbol{x}, M\Phi^{-s}\big)\, d\boldsymbol{x}\, d\mu_H(M)\, \mathrm{e}^{-d(d-1)s} ds. \end{aligned} \tag{3.50}$$

The asymptotics (3.9) show that

$$\limsup_{t\to\infty} \frac{|\mathcal{F}_Q \setminus \mathcal{F}_{Q,\theta}|}{\mathrm{e}^{d(d-1)t}} \leq \frac{\theta^d}{d\,\zeta(d)}, \tag{3.51}$$

which allows us to take the limit $\theta \to 0$ in (3.50). This concludes the proof for $\sigma = 0$ and f compactly supported. For the general case, recall the remarks in Step 0. □

Remark 3.3. Let $(\boldsymbol{p}, q) \in \widehat{\mathbb{Z}}$. Using the bijection (3.18), choose $\gamma \in \Gamma$ such that $(\boldsymbol{p}, q)\gamma = (\boldsymbol{0}, 1)$. For $\boldsymbol{r} = \boldsymbol{p}/q \in \mathcal{F}_Q + \mathbb{Z}^{d-1}$, we then have

$$\gamma^{-1}\,{}^{\mathrm{t}}(n_-(\boldsymbol{r})D(q))^{-1} = \left(\begin{pmatrix} q^{-1/(d-1)} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{p} & q \end{pmatrix} \gamma \right)^{-1} \in H. \tag{3.52}$$

That is,

$$\Gamma\,{}^{\mathrm{t}}(n_-(\boldsymbol{r})D(q))^{-1} \in \Gamma\backslash\Gamma H, \tag{3.53}$$

and thus, for $Q = \mathrm{e}^{(d-1)(t-\sigma)}$,

$$\Gamma\,{}^{\mathrm{t}}(n_-(\boldsymbol{r})\Phi^t)^{-1} \in \Gamma\backslash\Gamma H\{\Phi^{-s} : s \in \mathbb{R}_{\geq\sigma}\}. \tag{3.54}$$

Lemma 2. *The set* $\Gamma\backslash\Gamma H\{\Phi^{-s} : s \in \mathbb{R}_{\geq\sigma}\}$ *is a closed embedded submanifold of* $\Gamma\backslash G$.

Proof. The set

$$\Gamma\backslash\Gamma H\{\Phi^{-s} : s \in \mathbb{R}_{\geq\sigma}\} = \Gamma\backslash\Gamma H\{D(y_d) : y_d \in (0, c]\}, \qquad c = \mathrm{e}^{-(d-1)\sigma}, \tag{3.55}$$

is the image of the immersion map

$$i : \mathcal{H}_0 \to \Gamma\backslash G, \qquad \Gamma_H M \mapsto \Gamma M, \tag{3.56}$$

$$\mathcal{H}_0 := \Gamma_H\backslash H\{D(y_d) : y_d \in (0, c]\}, \tag{3.57}$$

and is thus an immersed submanifold of $\Gamma\backslash G$. To show that it is in fact a closed embedded submanifold, we need to establish that i is a proper map, i.e., every compact $\mathcal{K} \subset \Gamma\backslash G$ has a compact pre-image $i^{-1}(\mathcal{K})$; see e.g. [6, Chapter III]. Since i is continuous, $i^{-1}(\mathcal{K})$ is closed. It therefore suffices to show that $i^{-1}(\mathcal{K})$ is contained in a compact subset of $\mathcal{H}_0$.

For $M \in G$, let $I(M) = \inf\{\|\boldsymbol{m}M\| : \boldsymbol{m} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}\}$. By Mahler's criterion, there is $\theta > 0$ such that $I(M) \geq \theta$ for all $M \in G$ with $\Gamma M \in \mathcal{K}$. If $\Gamma_H M \in i^{-1}(\mathcal{K})$, then $I(M) \geq \theta$ with $M = hD(y_d)$, $h \in H$. Thus $(\boldsymbol{0}, 1)M = y_d$ and therefore $y_d \geq \theta$. This implies that, for any $h \in H$,

$$i(\Gamma_H h) = \Gamma h \in \mathcal{K}' := \mathcal{K}\{D(y_d)^{-1} : \theta \leq y_d \leq c\}, \tag{3.58}$$

where $\mathcal{K}'$ is a compact subset of $\Gamma\backslash G$.

It is a basic fact that, since H is a closed subgroup of G and $\Gamma_H = \Gamma \cap H$ is a lattice in H, the set $\Gamma\backslash\Gamma H$ is a closed embedded submanifold of $\Gamma\backslash G$ [12, Theorem 1.13]. We denote by $j : \Gamma_H\backslash H \to \Gamma\backslash\Gamma H$ the immersion map. Thus $j^{-1}(\mathcal{K}')$ is a compact subset of $\Gamma_H\backslash H$, and $i^{-1}(\mathcal{K})$ is contained in the compact subset $j^{-1}(\mathcal{K}')\{D(y_d) : \theta \leq y_d \leq c\}$ of $\mathcal{H}_0$. □

The significance of (3.54) and Lemma 2 is that it allows us reduce the continuity hypotheses of Theorem 6 and Remark 3.2 to continuity of $\widetilde{f}$ restricted to the closed embedded submanifold

$$\mathbb{T}^{d-1} \times \Gamma\backslash\Gamma H\{\Phi^{-s} : s \in \mathbb{R}_{\geq\sigma}\}. \tag{3.59}$$

We will exploit this fact in the proof of Theorem 8.

4. A variant of Theorem 6

The following variant of Theorem 6 will be key in the proof of Theorem 1. Recall the definition of $\widehat{\boldsymbol{a}}$ and $D(T)$ in (2.22) and (3.41), respectively.

Theorem 7. *Let $\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : 0 < x_1, \ldots, x_{d-1} \le x_d\}$ be bounded with boundary of Lebesgue measure zero, and $f : \overline{\mathcal{D}} \times \Gamma\backslash G \to \mathbb{R}$ bounded continuous. Then*

$$\lim_{T\to\infty} \frac{1}{T^d} \sum_{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d\cap T\mathcal{D}} f\left(\frac{\boldsymbol{a}}{T}, n_-(\widehat{\boldsymbol{a}})D(T)\right) = \frac{1}{\zeta(d)} \int_{\mathcal{D}\times\Gamma_H\backslash H} \widetilde{f}\big(\boldsymbol{y}, MD(y_d)\big)\, d\boldsymbol{y}\, d\mu_H(M) \tag{4.1}$$

with $\widetilde{f}(\boldsymbol{x}, M) := f(\boldsymbol{x}, {}^{\mathrm{t}}M^{-1})$.

Proof. Let $g : \mathbb{R}^{d-1} \times \Gamma\backslash G \to \mathbb{R}$ be a bounded continuous function. We apply Theorem 6 with $T = \mathrm{e}^{(d-1)t}$, $c = \mathrm{e}^{-(d-1)\sigma}$, and the test function

$$f(\boldsymbol{x}, M) = \sum_{\boldsymbol{n}\in\mathbb{Z}^{d-1}} g(\boldsymbol{x}+\boldsymbol{n}, M)\chi_{[0,1]^{d-1}}(\boldsymbol{x}+\boldsymbol{n}). \tag{4.2}$$

Note that this sum has at most 2^{d-1} non-zero terms. The function $f(\boldsymbol{x}, M)$ is bounded everywhere, and continuous on $[(0,1)^{d-1} + \mathbb{Z}^{d-1}] \times \Gamma\backslash G$; hence Remark 3.2, together with the asymptotics (3.9), yield

$$\begin{aligned} &\lim_{T\to\infty} \frac{\zeta(d)}{T^d} \sum_{\substack{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d \\ 1\le a_1,\ldots,a_{d-1}\le a_d \\ a_d\le cT}} g\big(\widehat{\boldsymbol{a}}, n_-(\widehat{\boldsymbol{a}})D(T)\big) \\ &= (d-1)\int_\sigma^\infty \int_{[0,1]^{d-1}\times\Gamma_H\backslash H} \widetilde{g}(\boldsymbol{x}, M\Phi^{-s})\, d\boldsymbol{x}\, d\mu_H(M)\, \mathrm{e}^{-d(d-1)s} ds \\ &= \int_0^c \int_{[0,1]^{d-1}\times\Gamma_H\backslash H} \widetilde{g}\big(\boldsymbol{x}, MD(y_d)\big)\, d\boldsymbol{x}\, d\mu_H(M)\, y_d^{d-1} dy_d \end{aligned} \tag{4.3}$$

where we have substituted in the last step $y_d = \mathrm{e}^{-(d-1)s}$. So for any $0 \le b < c$ we have

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T^d} \sum_{\substack{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d \\ 1\le a_1,\ldots,a_{d-1}\le a_d \\ bT<a_d\le cT}} g\big(\widehat{\boldsymbol{a}}, n_-(\widehat{\boldsymbol{a}})D(T)\big) \\ &= \frac{1}{\zeta(d)} \int_b^c \int_{[0,1]^{d-1}\times\Gamma_H\backslash H} \widetilde{g}\big(\boldsymbol{x}, MD(y_d)\big)\, d\boldsymbol{x}\, d\mu_H(M)\, y_d^{d-1} dy_d, \end{aligned} \tag{4.4}$$

and hence for $h : \mathbb{R}^{d-1} \times \mathbb{R} \times \Gamma\backslash G \to \mathbb{R}$ continuous with support in $\mathbb{R}^{d-1} \times \mathcal{I} \times \Gamma\backslash G$ and $\mathcal{I} \subset \mathbb{R}_{\ge 0}$ bounded, we have

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T^d} \sum_{\substack{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d \\ 1\le a_1,\ldots,a_{d-1}\le a_d}} h\left(\widehat{\boldsymbol{a}}, \frac{a_d}{T}, n_-(\widehat{\boldsymbol{a}})D(T)\right) \\ &= \frac{1}{\zeta(d)} \int_{[0,1]^{d-1}\times\mathcal{I}\times\Gamma_H\backslash H} \widetilde{h}\big(\boldsymbol{x}, y_d, MD(y_d)\big)\, d\boldsymbol{x}\, y_d^{d-1} dy_d\, d\mu_H(M). \end{aligned} \tag{4.5}$$

We now take $h(\boldsymbol{x}, y_d, M) = \chi_{\mathcal{D}}(\boldsymbol{x}y_d, y_d)\, f((\boldsymbol{x}y_d, y_d), M)$ with f as in Theorem 7, and substitute $\boldsymbol{y}' = \boldsymbol{x}y_d$. Note that with this choice h is no longer continuous; but $\mathcal{D}$ has boundary of measure zero and thus Remark 3.2 applies. $\square$

Remark 3.3 and Theorem 7 now imply the following theorem. Given a bounded subset $\mathcal{D} \subset \mathbb{R}^d_{\ge 0}$, define

$$\mathcal{M}_{\mathcal{D}} = \big\{(\boldsymbol{y}, \Gamma\, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}) : (\boldsymbol{y}, \Gamma M) \in \overline{\mathcal{D}} \times \Gamma\backslash\Gamma H\big\}, \tag{4.6}$$

which, in view of Lemma 2, is a closed embedded submanifold of $\mathbb{R}^d \times \Gamma\backslash G$. The bijection

$$\overline{\mathcal{D}} \times \Gamma_H\backslash H \to \mathcal{M}_{\mathcal{D}}, \qquad (\boldsymbol{y}, \Gamma_H M) \mapsto (\boldsymbol{y}, \Gamma\, {}^{\mathrm{t}}M^{-1} D(y_d)^{-1}), \tag{4.7}$$

allows us to define a natural measure ν on $\mathcal{M}_{\mathcal{D}}$ as the pushforward of $\mathrm{vol} \times \mu_H$, where vol is Lebesgue measure on $\mathbb{R}^d$ and μ_H as defined in (3.4). In the following we understand the interior and closure of subsets of $\mathcal{M}_{\mathcal{D}}$ with respect to the topology of $\mathcal{M}_{\mathcal{D}}$.

Since $n_-(\widehat{\boldsymbol{a}})D(T) = n_-(\widehat{\boldsymbol{a}})D(a_d)D(a_d/T)^{-1}$, eq. (3.53) implies that

$$\left(\frac{\boldsymbol{a}}{T}, \Gamma n_-(\widehat{\boldsymbol{a}})D(T)\right) \subset \mathcal{M}_{\mathcal{D}}. \tag{4.8}$$

Theorem 8. *Let $\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : 0 < x_1, \ldots, x_{d-1} \leq x_d\}$ be bounded with boundary of Lebesgue measure zero, and $\mathcal{A} \subset \mathcal{M}_{\mathcal{D}}$. Then*

$$\liminf_{T\to\infty} \frac{1}{T^d} \#\left\{\boldsymbol{a} \in \widehat{\mathbb{Z}}^d : \left(\frac{\boldsymbol{a}}{T}, \Gamma n_-(\widehat{\boldsymbol{a}})D(T)\right) \in \mathcal{A}\right\} \geq \frac{\nu(\mathcal{A}^\circ)}{\zeta(d)} \tag{4.9}$$

and

$$\limsup_{T\to\infty} \frac{1}{T^d} \#\left\{\boldsymbol{a} \in \widehat{\mathbb{Z}}^d : \left(\frac{\boldsymbol{a}}{T}, \Gamma n_-(\widehat{\boldsymbol{a}})D(T)\right) \in \mathcal{A}\right\} \leq \frac{\nu(\overline{\mathcal{A}})}{\zeta(d)}. \tag{4.10}$$

Proof. The inclusion (4.8) shows that the limit relation (4.1) in Theorem 7 holds for any bounded continuous function $f : \mathcal{M}_{\mathcal{D}} \to \mathbb{R}$. We can thus once more apply the above probabilistic argument [20, Chapter III] (used in the justification of Theorem 5) to prove (4.9) and (4.10). □

5. Upper and lower limits

Let us first of all note that we may assume in Theorem 1 without loss of generality that $\mathcal{D} \subset [0,1]^d$. Secondly, due to the symmetry of $F(\boldsymbol{a})$ under any permutation of the coefficients a_i, we may assume that $\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : 0 \leq x_1, \ldots, x_{d-1} \leq x_d\}$. Thirdly, it is sufficient to prove Theorem 1 for all bounded subsets of $\{\boldsymbol{x} \in \mathbb{R}^d : \eta \leq x_1, \ldots, x_{d-1} \leq x_d\}$, for any fixed $\eta > 0$. This is due to the fact that for any bounded set $\mathcal{D} \subset [0,1]^d$ with boundary of measure zero,

$$\lim_{T\to\infty} \frac{1}{T^d} \#\{\boldsymbol{a} \in \widehat{\mathbb{Z}}^d \cap T(\mathcal{D} \setminus \mathbb{R}^d_{\geq\eta})\} = \frac{\mathrm{vol}\,(\mathcal{D} \setminus \mathbb{R}^d_{\geq\eta})}{\zeta(d)} \leq \frac{d\,\eta}{\zeta(d)}. \tag{5.1}$$

We will therefore assume in the remainder of this section that, in addition to the assumptions of Theorem 1,

$$\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : \eta \leq x_1, \ldots, x_{d-1} \leq x_d \leq 1\}, \tag{5.2}$$

for arbitrary fixed $\eta > 0$.

The following is an immediate corollary of Theorem 3 (set $T = \mathrm{e}^{(d-1)t}$ and recall that $W_\delta(\lambda\boldsymbol{\alpha}, M) = \lambda W_\delta(\boldsymbol{\alpha}, M)$ for any $\lambda > 0$).

Lemma 3. *Let $\boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D}$ with $\mathcal{D}$ as in* (5.2)*, and $0 < \delta \leq \frac{1}{2}$. Then*

$$\left| \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} - \frac{W_\delta(\boldsymbol{y}', n_-(\widehat{\boldsymbol{a}})D(T))}{(y_1 \cdots y_d)^{1/(d-1)}} \right| \leq \frac{d}{\eta\, T^{1/(d-1)}}, \tag{5.3}$$

where $\boldsymbol{y} = T^{-1}\boldsymbol{a}$.

In view of this lemma, the plan is thus to apply Theorem 8 with the set

$$\mathcal{A} = \mathcal{A}_R = \left\{ (\boldsymbol{y}, \Gamma\, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}) : \boldsymbol{y} \in \mathcal{D},\ M \in \Gamma\backslash\Gamma H,\ \frac{W_\delta(\boldsymbol{y}', {}^{\mathrm{t}}M^{-1}D(y_d)^{-1})}{(y_1 \cdots y_d)^{1/(d-1)}} > R \right\}. \tag{5.4}$$

In the following, let

$$M = \begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} \in H, \tag{5.5}$$

where $A \in G_0$, $\boldsymbol{b} \in \mathbb{R}^{d-1}$. Then

$$ {}^{\mathrm{t}}M^{-1} = \begin{pmatrix} {}^{\mathrm{t}}A^{-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ -\boldsymbol{b}\,{}^{\mathrm{t}}A^{-1} & 1 \end{pmatrix}, \tag{5.6} $$

and

$$ (\boldsymbol{m}+\boldsymbol{\xi})\,{}^{\mathrm{t}}M^{-1}D(y_d)^{-1} = \big((\boldsymbol{m}'+\boldsymbol{\xi}'-(m_d+\xi_d)\boldsymbol{b})\,{}^{\mathrm{t}}A^{-1}y_d^{1/(d-1)}, (m_d+\xi_d)y_d^{-1}\big). \tag{5.7} $$

Assuming $\xi_d \in (-\frac{1}{2}, \frac{1}{2}]$, we deduce that, for all $0 < \delta \leq \frac{1}{2}$, the statement $(m_d+\xi_d)y_d^{-1} \in (-\delta, \delta)$ implies $m_d = 0$ since $0 < y_d \leq 1$. Therefore,

$$ \begin{aligned} &W_\delta(\boldsymbol{\alpha}, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}) \\ &= \sup_{\boldsymbol{\xi}\in\mathbb{T}^d} \min\nolimits_+ \big\{(\boldsymbol{m}+\boldsymbol{\xi})\,{}^{\mathrm{t}}M^{-1}D(y_d)^{-1}\cdot(\boldsymbol{\alpha},0) : \boldsymbol{m}\in\mathbb{Z}^d,\ (\boldsymbol{m}+\boldsymbol{\xi})\,{}^{\mathrm{t}}M^{-1}D(y_d)^{-1} \in \mathcal{R}_\delta\big\} \\ &= y_d^{1/(d-1)} \sup_{\substack{\boldsymbol{\xi}'\in\mathbb{T}^{d-1} \\ \xi_d\in(-\delta y_d,\delta y_d)}} \min\nolimits_+ \big\{(\boldsymbol{m}'+\boldsymbol{\xi}'-\xi_d\boldsymbol{b})\,{}^{\mathrm{t}}A^{-1}\cdot\boldsymbol{\alpha} : \boldsymbol{m}'\in\mathbb{Z}^{d-1},\ (\boldsymbol{m}'+\boldsymbol{\xi}'-\xi_d\boldsymbol{b})\,{}^{\mathrm{t}}A^{-1} \in \mathbb{R}^{d-1}_{\geq 0}\big\}. \end{aligned} \tag{5.8} $$

The substitution $\boldsymbol{\xi}' \mapsto \boldsymbol{\xi}' + \xi_d\boldsymbol{b}$ explains that the above supremum is independent of $\boldsymbol{b}$. So

$$ \begin{aligned} &W_\delta(\boldsymbol{\alpha}, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}) \\ &= y_d^{1/(d-1)} \sup_{\boldsymbol{\xi}'\in\mathbb{T}^{d-1}} \min\nolimits_+ \big\{(\boldsymbol{m}'+\boldsymbol{\xi}')\,{}^{\mathrm{t}}A^{-1}\cdot\boldsymbol{\alpha} : \boldsymbol{m}'\in\mathbb{Z}^{d-1},\ (\boldsymbol{m}'+\boldsymbol{\xi}')\,{}^{\mathrm{t}}A^{-1}\in\mathbb{R}^{d-1}_{\geq 0}\big\} \\ &= y_d^{1/(d-1)} V(\boldsymbol{\alpha}, {}^{\mathrm{t}}A^{-1}), \end{aligned} \tag{5.9} $$

where

$$ \begin{aligned} V(\boldsymbol{\alpha}, A) &= \sup_{\boldsymbol{\zeta}\in\mathbb{T}^{d-1}} \min\nolimits_+ \big\{(\boldsymbol{n}+\boldsymbol{\zeta})A\cdot\boldsymbol{\alpha} : \boldsymbol{n}\in\mathbb{Z}^{d-1},\ (\boldsymbol{n}+\boldsymbol{\zeta})A\in\mathbb{R}^{d-1}_{\geq 0}\big\} \\ &= \sup_{\boldsymbol{\zeta}\in\mathbb{T}^{d-1}} \min\nolimits_+ \big((\mathbb{Z}^{d-1}+\boldsymbol{\zeta})A \cap \mathbb{R}^{d-1}_{\geq 0}\big)\cdot\boldsymbol{\alpha}. \end{aligned} \tag{5.10} $$

Now set $\boldsymbol{\alpha} = \boldsymbol{y}' = (y_1, \ldots, y_{d-1})$, and

$$ Y = (y_1\cdots y_{d-1})^{-1/(d-1)}\,\mathrm{diag}(y_1,\ldots,y_{d-1}) \in G_0, \tag{5.11} $$

so that $\boldsymbol{y}' = (y_1\cdots y_{d-1})^{1/(d-1)}\boldsymbol{e}Y$. Then

$$ V(\boldsymbol{y}', A) = (y_1\cdots y_{d-1})^{1/(d-1)} V(\boldsymbol{e}, AY) \tag{5.12} $$

and hence

$$ \frac{W_\delta(\boldsymbol{y}', {}^{\mathrm{t}}M^{-1}D(y_d)^{-1})}{(y_1\cdots y_d)^{1/(d-1)}} = V(\boldsymbol{e}, {}^{\mathrm{t}}A^{-1}Y). \tag{5.13} $$

Set

$$ V(A) := V(\boldsymbol{e}, A) = \sup_{\boldsymbol{\zeta}\in\mathbb{T}^{d-1}} \min\big((\mathbb{Z}^{d-1}+\boldsymbol{\zeta})A\cap\mathbb{R}^{d-1}_{\geq 0}\big)\cdot\boldsymbol{e}. \tag{5.14} $$

We conclude that

$$ \mathcal{A}_R = \Big\{\big(\boldsymbol{y}, \Gamma\,{}^{\mathrm{t}}M^{-1}D(y_d)^{-1}\big) : (\boldsymbol{y}, M) \in \mathcal{D}\times\Gamma\backslash\Gamma H,\ V({}^{\mathrm{t}}A^{-1}Y) > R\Big\}. \tag{5.15} $$

Lemma 4. *$V(A)$ is a continuous function on $\Gamma_0\backslash G_0$.*

Proof. We have $V(\gamma A) = V(A)$ for all $\gamma \in \Gamma_0$ by the same argument as in (2.19), and hence $V(A)$ is a function on $\Gamma_0\backslash G_0$. It is sufficient to establish the continuity of $V(A)$ on compact subsets of G_0. Let us thus fix a compact set $\mathcal{C} \subset G_0$, and define

$$ K = \{\boldsymbol{\zeta}A : \boldsymbol{\zeta}\in[0,1]^{d-1},\ A\in\mathcal{C}\}, \tag{5.16} $$

which is a compact subset of $\mathbb{R}^{d-1}$. Then, for all $A \in \mathcal{C}$,

$$V(A) = \sup_{\boldsymbol{x}\in L} \min \big((\mathbb{Z}^{d-1}A + \boldsymbol{x}) \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e}, \tag{5.17}$$

where L is any set containing K. Clearly $V(A)$ is bounded on $\mathcal{C}$, i.e., there is $R > 0$ such that $V(A) \leq R$ for all $A \in \mathcal{C}$. Thus

$$V(A) = \sup_{\boldsymbol{x}\in L} \min \big((\mathbb{Z}^{d-1}A + \boldsymbol{x}) \cap R\Delta \big) \cdot \boldsymbol{e}, \tag{5.18}$$

where Δ is the simplex (1.5). For $K' = K + [-1, 1]\boldsymbol{e}$,

$$S = \mathbb{Z}^{d-1} \cap \bigcup_{A\in\mathcal{C}} \bigcup_{\boldsymbol{x}\in K'} \big((R\Delta - \boldsymbol{x})A^{-1} \big) \tag{5.19}$$

is a finite subset of $\mathbb{Z}^{d-1}$, and we have

$$V(A) = \sup_{\boldsymbol{x}\in K'} \min_{\boldsymbol{m}\in S} \big((\boldsymbol{m}A + \boldsymbol{x}) \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e} \tag{5.20}$$

for all $A \in \mathcal{C}$. (The reason why we use K' rather than K in the definition of S will become clear below.)

Fix $\epsilon \in (0, 1)$. Then there exists $\delta > 0$ such that, for all $A, A' \in \mathcal{C}$ with $d(A, A') < \delta$, we have

$$\|\boldsymbol{m}A - \boldsymbol{m}A'\| < \epsilon \qquad \text{for all } \boldsymbol{m} \in S. \tag{5.21}$$

Thus, for any $\boldsymbol{m} \in S$ we have

$$\boldsymbol{m}A' + \boldsymbol{x} - \epsilon\boldsymbol{e} \in \mathbb{R}^{d-1}_{\geq 0} \quad \text{implies} \quad \boldsymbol{m}A + \boldsymbol{x} \in \mathbb{R}^{d-1}_{\geq 0}, \tag{5.22}$$

and secondly

$$\begin{aligned} (\boldsymbol{m}A' + \boldsymbol{x} - \epsilon\boldsymbol{e}) \cdot \boldsymbol{e} &= (\boldsymbol{m}A' + \boldsymbol{x}) \cdot \boldsymbol{e} - d\epsilon \\ &\geq (\boldsymbol{m}A + \boldsymbol{x}) \cdot \boldsymbol{e} - (\sqrt{d} + d)\epsilon. \end{aligned} \tag{5.23}$$

Now choose $\boldsymbol{x} \in K$ such that

$$\min \big((\mathbb{Z}^{d-1}A + \boldsymbol{x}) \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e} \geq V(A) - \epsilon. \tag{5.24}$$

Then (5.22) and (5.23) yield

$$\min_{\boldsymbol{m}\in S} \big((\boldsymbol{m}A' + \boldsymbol{x} - \epsilon\boldsymbol{e}) \cap \mathbb{R}^{d-1}_{\geq 0} \big) \cdot \boldsymbol{e} \geq V(A) - (1 + \sqrt{d} + d)\epsilon. \tag{5.25}$$

Since $\boldsymbol{x} - \epsilon\boldsymbol{e} \in K'$ (because $\boldsymbol{x} \in K$ and $0 < \epsilon < 1$), the left hand side is at most $V(A')$. That is, $V(A') \geq V(A) - (1 + \sqrt{d} + d)\epsilon$. We conclude by interchanging A and A' that

$$|V(A') - V(A)| \leq (1 + \sqrt{d} + d)\epsilon. \tag{5.26}$$

for all $A, A' \in \mathcal{C}$ with $d(A, A') < \delta$. □

Since $V(A)$ is continuous, we have for any $\epsilon \in (0, R]$,

$$\mathcal{A}_{R+\epsilon} \subset \mathcal{A}_R^\circ \subset \overline{\mathcal{A}}_R \subset \mathcal{A}_{R-\epsilon}. \tag{5.27}$$

Define the function $\Psi_d : \mathbb{R}_{\geq 0} \to [0, 1]$ by

$$\Psi_d(R) := \mu_0\big(\{A \in \Gamma_0\backslash G_0 : V(A) > R\}\big), \tag{5.28}$$

which is non-increasing. Note that by the invariance of μ_0 under the right G_0-action and under $A \mapsto {}^{\mathrm{t}}A^{-1}$, we have

$$\Psi_d(R) = \mu_0\big(\{A \in \Gamma_0\backslash G_0 : V({}^{\mathrm{t}}A^{-1}Y) > R\}\big). \tag{5.29}$$

As to the right hand sides of (4.9) and (4.10), the above calculations show that for any $\epsilon \in (0, R]$,

$$\nu\big(\mathcal{A}_R^\circ\big) \geq \mathrm{vol}(\mathcal{D})\, \Psi_d(R + \epsilon) \tag{5.30}$$

and

$$\nu(\overline{\mathcal{A}}_R) \leq \operatorname{vol}(\mathcal{D})\, \Psi_d(R-\epsilon). \tag{5.31}$$

Thus, combining these inequalities with Theorem 8 and Lemma 3, we obtain the following.

Lemma 5. *Let $R > 0$. For any $\epsilon \in (0, R]$,*

$$\liminf_{T\to\infty} \frac{1}{T^d} \#\left\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \right\} \geq \frac{\operatorname{vol}(\mathcal{D})}{\zeta(d)}\, \Psi_d(R+\epsilon), \tag{5.32}$$

$$\limsup_{T\to\infty} \frac{1}{T^d} \#\left\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d_{\geq 2} \cap T\mathcal{D} : \frac{F(\boldsymbol{a})}{(a_1 \cdots a_d)^{1/(d-1)}} > R \right\} \leq \frac{\operatorname{vol}(\mathcal{D})}{\zeta(d)}\, \Psi_d(R-\epsilon). \tag{5.33}$$

With this lemma, the proof of Theorem 1 is complete if we can show that $\Psi_d(R)$ is continuous (since then the lim sup and lim inf must coincide). This will be proved in Section 7.

6. Lattice free domains and covering radii

We denote the standard basis vectors in $\mathbb{R}^{d-1}$ by $\boldsymbol{e}_1 = (1, 0, \ldots, 0), \ldots, \boldsymbol{e}_{d-1} = (0, \ldots, 0, 1)$. Consider the simplex (1.5) and denote the face perpendicular to $\boldsymbol{e}_i$ by Δ_i ($i = 1, \ldots, d-1$), and by Δ_d the face perpendicular to $\boldsymbol{e}$.

Recall from the previous section:

$$V(A) = \sup_{\boldsymbol{\zeta}\in\mathbb{T}^{d-1}} \min\left((\mathbb{Z}^{d-1} + \boldsymbol{\zeta})A \cap \mathbb{R}^{d-1}_{\geq 0} \right) \cdot \boldsymbol{e}. \tag{6.1}$$

The following lemma states, that the simplex Δ, enlarged by a factor of $V(A)$ and suitably translated, is a maximal lattice free domain; cf. also [16].

Lemma 6. *If $V(A) = R$ for some $R > 0$, then there is a vector $\boldsymbol{\zeta} \in \mathbb{R}^{d-1}$ such that*

(i) $\mathbb{Z}^{d-1}A \cap (R\Delta^\circ + \boldsymbol{\zeta}) = \emptyset$;

(ii) $\mathbb{Z}^{d-1}A \cap (R\Delta_i^\circ + \boldsymbol{\zeta}) \neq \emptyset$ *for all $i = 1, \ldots, d$.*

On the other hand, if (i) *and* (ii) *hold for some $R > 0$, $\boldsymbol{\zeta} \in \mathbb{R}^{d-1}$, then $R \leq V(A)$.*

Proof. If $\mathbb{Z}^{d-1}A \cap (R\Delta^\circ + \boldsymbol{\zeta}) \neq \emptyset$ for all $\boldsymbol{\zeta}$, then $V(A) < R$, contradicting our assumption $V(A) = R$. Hence there exists $\boldsymbol{\zeta}$ such that (i) holds. If $\mathbb{Z}^{d-1}A \cap (R\Delta_i^\circ + \boldsymbol{\zeta}) = \emptyset$ for some i, then there exists a larger translate $R'\Delta^\circ + \boldsymbol{\zeta}'$ (for some $R' > R$, $\boldsymbol{\zeta}' \in \mathbb{R}^{d-1}$) which is lattice free, and hence $V(A) \geq R' > R$. This proves (ii), and the final statement is evident. □

Theorem 9. *Denote by $\rho(A)$ the covering radius of the simplex Δ with respect to the lattice $\mathbb{Z}^{d-1}A$. Then*

$$\rho(A) = V(A). \tag{6.2}$$

Proof. (We adapt the argument of [16, Theorem 2].) Let $V(A) = R$ and assume $\mathbb{Z}^{d-1}A + R\Delta \neq \mathbb{R}^{d-1}$. Then there is $\boldsymbol{\xi} \in \mathbb{R}^{d-1}$ such that $\boldsymbol{\xi} + \boldsymbol{v} \notin R\Delta$ for all $\boldsymbol{v} \in \mathbb{Z}^{d-1}A$. Hence $\mathbb{Z}^{d-1}A \cap (R\Delta - \boldsymbol{\xi}) = \emptyset$, and, by Lemma 6, $V(A) > R$; a contradiction. This shows $\rho(A) \leq V(A)$.

On the other hand, again by Lemma 6, for any $R' < R = V(A)$ there exists $\boldsymbol{\zeta} \in \mathbb{R}^{d-1}$ such that $\mathbb{Z}^{d-1}A \cap (R'\Delta + \boldsymbol{\zeta}) = \emptyset$, and hence no element of $\mathbb{Z}^{d-1}A$ is covered by the translates of $R'\Delta + \boldsymbol{\zeta}$. This proves $\rho(A) > R'$ and hence $\rho(A) = V(A)$. □

7. Continuity of the limit distribution

The following lemma shows that $\Psi_d(R)$ is continuous.

Lemma 7. *For every $R > 0$,*

$$\mu_0\left(\{ A \in \Gamma_0 \backslash G_0 : V(A) = R \} \right) = 0. \tag{7.1}$$

Proof. By Lemma 6 (ii), the set $\{A \in G_0 : V(A) = R\}$ is a subset of

(7.2)
$$\bigcup_{\boldsymbol{n}_1,\ldots,\boldsymbol{n}_d \in \mathbb{Z}^{d-1}} \{A \in G_0 : \text{there exists } \boldsymbol{\zeta} \in \mathbb{R}^{d-1} \text{ such that } \boldsymbol{n}_i A \cap (R\Delta_i^\circ + \boldsymbol{\zeta}) \neq \emptyset \ (i = 1, \ldots, d)\}.$$

We therefore need to show that each set in the above union has μ_0-measure zero. Since the sets $R\Delta_i^\circ$ are contained in the respective hyperplanes $\boldsymbol{e}_i \cdot \boldsymbol{y} = 0$ (for $i = 1, \ldots, d-1$) and $\boldsymbol{e} \cdot \boldsymbol{y} = R$ (for $i = d$), it suffices to show that

$$\{A \in G_0 : \text{there exists } \boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_{d-1}) \in \mathbb{R}^{d-1} \text{ such that } \boldsymbol{e}_i \cdot \boldsymbol{n}_i A = \zeta_i \ (i = 1, \ldots, d-1),\ \boldsymbol{e} \cdot \boldsymbol{n}_d A = R + \boldsymbol{e} \cdot \boldsymbol{\zeta}\} \tag{7.3}$$

has measure zero. Evidently (7.3) equals

$$\Big\{A \in G_0 : \boldsymbol{e} \cdot \boldsymbol{n}_d A = R + \sum_{i=1}^{d-1} \boldsymbol{e}_i \cdot \boldsymbol{n}_i A\Big\} = \Big\{A \in G_0 : \operatorname{tr}(LA) = R\Big\}, \tag{7.4}$$

with the matrix

$$L = \begin{pmatrix} \boldsymbol{n}_d - \boldsymbol{n}_1 \\ \vdots \\ \boldsymbol{n}_d - \boldsymbol{n}_{d-1} \end{pmatrix}. \tag{7.5}$$

If $L = 0$ the set (7.4) is empty (since $R > 0$) and hence has measure zero. If $L \neq 0$ then the set (7.4) is a submanifold of codimension one; note that the map $G_0 \to \mathbb{R}$, $A \mapsto \operatorname{tr}(LA)$, has non-vanishing differential except at the (at most two) points $A \in G_0$ for which LA is proportional to the identity matrix. Hence the set (7.4) has measure zero also in this case and the proof is complete. □

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Appendix A. The distribution of sublattices

Sections 3 and 4 establish the equidistribution of Farey sequences embedded in large horospheres. These results provide an alternative perspective on Schmidt's work on the distribution of sublattices of $\mathbb{Z}^d$ [17]. In the present appendix, we will reformulate Theorems 7 and 8 in a form that clarifies the relationship between the two approaches.

Let us fix a piecewise continuous map $K : \mathrm{S}_1^{d-1} \to G$ of the unit sphere S_1^{d-1} such that $\boldsymbol{y}K(\boldsymbol{y}) = (\boldsymbol{0}, 1)$. By *piecewise continuous* we mean here: there is a partition of S_1^{d-1} by subsets $\mathcal{P}_i$ with boundary of Lebesgue measure zero, so that K restricted to $\mathcal{P}_i$ can be extended to a continuous map on the closure $\overline{\mathcal{P}}_i$.

We extend the definition of K to $\mathbb{R}^d \setminus \{\boldsymbol{0}\} \to G$ by setting

$$K(\boldsymbol{y}) = K(\mathring{\boldsymbol{y}}) D(\|\boldsymbol{y}\|)^{-1} \tag{A.1}$$

with D as in (3.41) and $\mathring{\boldsymbol{y}} := \boldsymbol{y}/\|\boldsymbol{y}\|$. The extended map still satisfies $\boldsymbol{y}K(\boldsymbol{y}) = (\boldsymbol{0}, 1)$.

As in Remark 3.3, we choose $\gamma \in \Gamma$ such that $\boldsymbol{a}\gamma = (\boldsymbol{0}, 1)$. Then $(\boldsymbol{0}, 1)\gamma^{-1}K(\boldsymbol{a}) = (\boldsymbol{0}, 1)$, which implies $\gamma^{-1}K(\boldsymbol{a}) \in H$, and hence $\Gamma K(\boldsymbol{a}) \in \Gamma\backslash\Gamma H$.

Theorem 10. *Fix a piecewise continuous embedding $K : \mathbb{R}^d \setminus \{\boldsymbol{0}\} \to G$ as defined above. Let $\mathcal{D} \subset \mathbb{R}^d$ be bounded with boundary of Lebesgue measure zero, and $f : \overline{\mathcal{D}} \times \Gamma\backslash\Gamma H \to \mathbb{R}$ bounded continuous. Then*

$$\lim_{T\to\infty} \frac{1}{T^d} \sum_{\boldsymbol{a} \in \widehat{\mathbb{Z}}^d \cap T\mathcal{D}} f\Big(\frac{\boldsymbol{a}}{T}, K(\boldsymbol{a})\Big) = \frac{1}{\zeta(d)} \int_{\mathcal{D}\times\Gamma_H\backslash H} f\big(\boldsymbol{y}, M\big)\, d\boldsymbol{y}\, d\mu_H(M). \tag{A.2}$$

Proof. In view of the fact that $\Gamma\backslash\Gamma H$ is a closed embedded submanifold of $\Gamma\backslash G$, it suffices to prove that, for $f : \overline{\mathcal{D}} \times \Gamma\backslash G \to \mathbb{R}$ bounded continuous,

$$\lim_{T\to\infty} \frac{1}{T^d} \sum_{\boldsymbol{a}\in\widehat{\mathbb{Z}}^d\cap T\mathcal{D}} f\Big(\frac{\boldsymbol{a}}{T}, {}^{\mathrm{t}}K(\boldsymbol{a})^{-1}\Big) = \frac{1}{\zeta(d)} \int_{\mathcal{D}\times\Gamma_H\backslash H} \widetilde{f}\big(\boldsymbol{y}, M\big)\, d\boldsymbol{y}\, d\mu_H(M). \tag{A.3}$$

We may assume without loss of generality that f has compact support (cf. Step 0 of the proof of Theorem 6), and that $\mathcal{D} \subset \{\boldsymbol{x} \in \mathbb{R}^d : \eta \le x_1, \ldots, x_{d-1} \le x_d\} \cap \mathbb{R}_{>0}\mathcal{P}_i$ for some fixed $\eta > 0$ and $\mathcal{P}_i$ as defined in the second paragraph of this appendix.

If $\boldsymbol{y} \in \mathcal{D}$, then $y_d \ge \eta$, and we may expand

$$K(\mathring{\boldsymbol{y}})^{-1} = \begin{pmatrix} A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{b}(\mathring{\boldsymbol{y}}) \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\mathring{\boldsymbol{y}}}, \tag{A.4}$$

with $M_{\boldsymbol{y}}$ as in (3.21). The maps A, $\boldsymbol{b}$ are continuous on $\overline{\mathcal{P}}_i \cap \mathbb{R}^d_{\ge\eta}$, and hence bounded. A short calculation shows that

$$\begin{aligned} K(\boldsymbol{y})^{-1} &= \begin{pmatrix} A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{b}(\mathring{\boldsymbol{y}})\|\boldsymbol{y}\|^{-d/(d-1)} \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\boldsymbol{y}} \\ &= \begin{pmatrix} 1_{d-1} & {}^{\mathrm{t}}\boldsymbol{b}(\mathring{\boldsymbol{y}})\|\boldsymbol{y}\|^{-d/(d-1)} \\ \boldsymbol{0} & 1 \end{pmatrix} \begin{pmatrix} A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\boldsymbol{y}}. \end{aligned} \tag{A.5}$$

Set

$$K_0(\boldsymbol{y})^{-1} = \begin{pmatrix} A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix} M_{\boldsymbol{y}}. \tag{A.6}$$

Because $\|\boldsymbol{a}\| \ge \sqrt{d}\,\eta T$, we have

$$d({}^{\mathrm{t}}K(\boldsymbol{a})^{-1}, {}^{\mathrm{t}}K_0(\boldsymbol{a})^{-1}) \le \sup_{\boldsymbol{y}\in\mathcal{D}} \big\|\boldsymbol{b}(\mathring{\boldsymbol{y}})\big\| \,\big(\sqrt{d}\,\eta T\big)^{-d/(d-1)}, \tag{A.7}$$

where the supremum is finite by the continuity of $\boldsymbol{b}$. Since f is uniformly continuous, it therefore suffices to establish (A.3) with $K(\boldsymbol{a})^{-1}$ replaced by $K_0(\boldsymbol{a})^{-1}$. We now apply Theorem 7 with the test function

$$f_0(\boldsymbol{y}, M) = f\bigg(\boldsymbol{y}, M D(y_d) \begin{pmatrix} {}^{\mathrm{t}}A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}\bigg), \tag{A.8}$$

which is bounded continuous on $\overline{\mathcal{D}} \times \Gamma\backslash G$ (under the above assumptions on f and $\mathcal{D}$). With this choice,

$$\begin{aligned} f_0\Big(\frac{\boldsymbol{a}}{T}, n_-(\widehat{\boldsymbol{a}})D(T)\Big) &= f\bigg(\frac{\boldsymbol{a}}{T}, n_-(\widehat{\boldsymbol{a}})D(T)D(a_d/T) \begin{pmatrix} {}^{\mathrm{t}}A(\mathring{\boldsymbol{a}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}\bigg) \\ &= f\Big(\frac{\boldsymbol{a}}{T}, {}^{\mathrm{t}}K_0(\boldsymbol{a})^{-1}\Big). \end{aligned} \tag{A.9}$$

As to the right hand side of (4.1), we have

$$\begin{aligned} \widetilde{f}_0(\boldsymbol{y}, MD(y_d)) &= f_0\big(\boldsymbol{y}, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}\big) \\ &= f\bigg(\boldsymbol{y}, {}^{\mathrm{t}}M^{-1}D(y_d)^{-1}D(y_d) \begin{pmatrix} {}^{\mathrm{t}}A(\mathring{\boldsymbol{y}}) & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}\bigg) \\ &= \widetilde{f}\bigg(\boldsymbol{y}, M \begin{pmatrix} A(\mathring{\boldsymbol{y}})^{-1} & {}^{\mathrm{t}}\boldsymbol{0} \\ \boldsymbol{0} & 1 \end{pmatrix}\bigg). \end{aligned} \tag{A.10}$$

Eq. (A.3) now follows from the right H-invariance of μ_H. □

The following theorem is a corollary of Theorem 10; the proof is analogous to that of Theorem 8.

Theorem 11. *Fix a piecewise continuous embedding $K : \mathbb{R}^d \setminus \{\mathbf{0}\} \to G$ as defined above. Let $\mathcal{D} \subset \mathbb{R}^d$ be bounded with boundary of Lebesgue measure zero, and $\mathcal{A} \subset \overline{\mathcal{D}} \times \Gamma\backslash\Gamma H$. Then*

$$\liminf_{T\to\infty} \frac{1}{T^d} \#\Big\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d : \Big(\frac{\boldsymbol{a}}{T}, \Gamma K(\boldsymbol{a}) \Big) \in \mathcal{A} \Big\} \geq \frac{(\operatorname{vol} \times \mu_H)(\mathcal{A}^\circ)}{\zeta(d)} \tag{A.11}$$

and

$$\limsup_{T\to\infty} \frac{1}{T^d} \#\Big\{ \boldsymbol{a} \in \widehat{\mathbb{Z}}^d : \Big(\frac{\boldsymbol{a}}{T}, \Gamma K(\boldsymbol{a}) \Big) \in \mathcal{A} \Big\} \leq \frac{(\operatorname{vol} \times \mu_H)(\overline{\mathcal{A}})}{\zeta(d)}. \tag{A.12}$$

Let us now explain how the above statements are related to Schmidt's results on the distribution of primitive sublattices [17].

Two lattices $\Lambda, \Lambda' \subset \mathbb{R}^d$ of rank m are called *similar*, if there is an invertible angle-preserving linear transformation R (that is, $R \in \mathbb{R}_{>0}\operatorname{O}(d)$), such that $\Lambda' = \Lambda R$.

Let us denote by $\operatorname{Gr}_m(\mathbb{R}^d)$ the Grassmannian of m-dimensional linear subspaces of $\mathbb{R}^d$. The map

$$\widehat{\mathbb{Z}}^d \to \operatorname{Gr}_{d-1}(\mathbb{R}^d), \qquad \boldsymbol{a} \mapsto \boldsymbol{a}^\perp := \{ \boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} \cdot \boldsymbol{a} = 0 \} \tag{A.13}$$

gives a one-to-one correspondence between primitive lattice points and rational subspaces of dimension $d-1$. A *primitive sublattice of* $\mathbb{Z}^d$ *of rank* $d-1$ is defined as

$$\Lambda_{\boldsymbol{a}} = \mathbb{Z}^d \cap \boldsymbol{a}^\perp, \tag{A.14}$$

and hence there is a one-to-one correspondence between primitive lattice points and primitive sublattices of rank $d-1$. The covolume of $\Lambda_{\boldsymbol{a}}$ equals $\|\boldsymbol{a}\|$. Note that

$$\boldsymbol{a}^\perp \, {}^{\mathrm{t}}K(\boldsymbol{a})^{-1} = (\mathbf{0}, 1)^\perp = \mathbb{R}^{d-1} \times \{0\}, \tag{A.15}$$

with $K(\boldsymbol{a})$ as in (A.1). Hence

$$\Lambda_{\boldsymbol{a}} \, {}^{\mathrm{t}}K(\boldsymbol{a})^{-1} = \mathbb{Z}^d \, {}^{\mathrm{t}}K(\boldsymbol{a})^{-1} \cap (\mathbb{R}^{d-1} \times \{0\}) \tag{A.16}$$

and

$$\Lambda_{\boldsymbol{a}} \, {}^{\mathrm{t}}K(\boldsymbol{a})^{-1} = \|\boldsymbol{a}\|^{-1/(d-1)} \Lambda_{\boldsymbol{a}} \, {}^{\mathrm{t}}K(\mathring{\boldsymbol{a}})^{-1}. \tag{A.17}$$

We now choose the above embedding K such that $K(\mathring{\boldsymbol{y}}) \in \operatorname{SO}(d)$; see e.g. [11, Section 4.2, footnote 3] for an explicit construction. The map

$$\Lambda_{\boldsymbol{a}} \mapsto \Lambda'_{\boldsymbol{a}} := \Lambda_{\boldsymbol{a}} \, {}^{\mathrm{t}}K(\boldsymbol{a})^{-1} \tag{A.18}$$

maps primitive sublattices of $\mathbb{Z}^d$ of rank $d-1$ to lattices in $\mathbb{R}^{d-1}$. Eq. (A.17) shows $\Lambda_{\boldsymbol{a}}$ and $\Lambda'_{\boldsymbol{a}}$ are similar; it furthermore implies that $\Lambda'_{\boldsymbol{a}}$ has covolume one.

In [17] Schmidt proves that, as $T \to \infty$, the set $\{\Lambda'_{\boldsymbol{a}} : \|\boldsymbol{a}\| \leq T\}$ becomes uniformly distributed in the space of lattices of covolume one, $\Gamma_0\backslash G_0$, with respect to the right G_0-invariant measure μ_0. In particular, Theorem 3 in [17] (adapted to the case of primitive lattices of rank $d-1$) follows from our Theorem 11, if we set

$$\mathcal{A} = \left\{ \left(\boldsymbol{y}, \Gamma \begin{pmatrix} A & {}^{\mathrm{t}}\boldsymbol{b} \\ \mathbf{0} & 1 \end{pmatrix} \right) : \boldsymbol{y} \in \mathcal{D},\ A \in \mathcal{A}_0,\ \boldsymbol{b} \in \mathbb{R}^{d-1} \right\} \subset \overline{\mathcal{D}} \times \Gamma\backslash\Gamma H, \tag{A.19}$$

where $\mathcal{D} \subset \mathbb{R}^d$ has boundary of Lebesgue measure zero, and $\mathcal{A}_0 \subset \Gamma_0\backslash G_0$ is arbitrary. Theorem 2 in [17] is obtained when $\mathcal{D}$ is taken to be the unit ball.

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